# UNDERSTANDING MATHEMATICS TO UNDERSTAND PLATO -THEAETEUS (147d-148b

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**UNDERSTANDING MATHEMATICS TO UNDERSTAND PLATO - *THEAETEUS* (147d-148b)**
***COMPRENDRE LES MATHÉMATIQUES POUR COMPRENDRE PLATON* - THÉÉTÈTE *(147d-148b)***

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**Abstract**. This paper is an updated translation of an article published in French in the Journal *Lato Sensu* (I, 2014, p. 70-80). We study here the so-called 'Mathematical part' of Plato's Theaetetus. Its subject concerns the incommensurability of certain magnitudes, in modern terms the question of the rationality or irrationality of the square roots of integers. As the most ancient text on the subject, and on Greek mathematics and mathematicians as well, its historical importance is enormous. The difficulty to understand it lies in the close intertwining of different fields we found in it: philosophy, history and mathematics. But conversely, correctly understood, it gives some evidences both about the question of the origins of the irrationals in Greek mathematics and some points concerning Plato's thought. Taking into account the historical context and the philosophical background generally forgotten in mathematical analyses, we get a new interpretation of this text, which far from being a tribute to some mathematicians, is a radical criticism of their ways of thinking. And the mathematical lesson, far from being a tribute to some future mathematical achievements, is ending on an aporia, in accordance with the whole dialogue.

**Résumé.** Cet article est une traduction anglaise révisée d'un article paru en français dans la revue *Lato Sensu* (I, 2014, p. 70-80). Il s'agit de l'étude des premières lignes de ce que l'on nomme traditionnellement la 'partie mathématique' du *Théétète* de Platon, où un jeune Athénien, Théétète, rapporte une leçon de mathématiques sur l'incommensurabilité de certaines grandeurs, à laquelle il a assisté. En termes modernes, il s'agit de la question de la rationalité (ou de l'irrationalité) des racines carrées des nombres entiers. En tant que le plus ancien texte qui nous soit parvenu sur le sujet, mais aussi sur les mathématiques et les mathématiciens grecs, sa valeur est inestimable. Les difficultés pour l'interpréter proviennent de l'étroite imbrication qu'on y trouve entre différents domaines : philosophie, histoire et mathématiques. Mais inversement, convenablement compris, il peut fournir des témoignages à la fois sur la question des origines de la théorie des irrationnels dans les mathématiques grecques et sur certains points de la pensée platonicienne. À partir d'une analyse mathématique prenant en compte le contexte historique et l'arrière-plan philosophique du dialogue généralement négligés, nous obtenons une interprétation nouvelle de ce texte qui, loin d'être un hommage à certains mathématiciens, est une critique radicale de leurs manières de penser. Et la leçon mathématique, loin d'être un hommage à de futurs succès mathématiques, apparaît, de manière cohérente avec le dialogue tout entier, se conclure sur une aporie.

**Keywords**: *anthyphairesis*, apory, Babylonian mathematics, definition, demonstration, irrationals (origin of the), knowledge, learning, mathematics, mathematical truth, philosophy, Plato, Pythagoras' theorem, science.

## 1. Présentation de l’ouvrage : le prologue

In Plato’s book, the dialogue is preceded by a prologue between two characters, Euclid[1] and Terpsion. It is taking place in Megara, Euclid having escorted Theaetetus dying after a battle near the town of Corinth[2]. He was not hurt during the battle but he is dying from disease, the dysentery, which has spread inside the Athenian army.

Since the two men have walked a long road, they decide to take a rest and Terpsion asks his companion to relate him a discussion Theaetetus participated. Socrates told it to Euclid who immediately wrote it, asking Socrates to correct it ‘every time he met him’[3].

The characters in the meeting are Theaetetus, Socrates, then very young, and Theodorus, a mathematician from Cyrene, a Greek settlement located in the north-east of actual Libya. Nevertheless, in the part of the dialogue we will study, only the two formers participate directly.

[1] He is Euclid of Megara confused for a long time with the mathematician Euclid of Alexandria, the author of the *Elements*.

[2] There is no consensus about the dating of this battle or rather to which battle Plato is referring. There are two possibilities, one around 390 BCE and another around 369 BCE (‘*Before Common Era*’).

[3] ‘… καὶ ὁσάκις ’Αθήναζε ἀφικοίμην, ἐπανηρώτων τὸν Σωκράτη ὅ μὴ ἐμεμνήμην’ (142d7). We can wonder if the number of these meetings (‘ὁσάκις’) was so great, since the meeting happened very little time before Socrates was sentenced to death (210d).

## 2. Introduction

At the beginning of the dialogue, Socrates is asking Theodorus some news about the young Athenian pupils who are learning mathematics with him. Theodorus answers by speaking eulogistically about one of them, the young Theaetetus. According to the mathematician, he is both extremely gifted, open minded and generous, but also very ugly, 'as much as Socrates' says Theodorus. Since by chance the young boy is just leaving the gymnasium near from them, on Socrates' request, Theodorus calls him. Thus begins the scene with the three characters.

Soon Socrates raises doubts about Theodorus' knowledge on physical beauty and ugliness, and Theaetetus is compelled, in spite of some reluctance, to agree with him.

Immediately afterwards, Socrates suggests the subject of a study: a discussion about/definition of (λόγος) what is science/knowledge (ἐπιστήμη)[4].

Asked to be the respondent, Theodorus refuses and proposes Theaetetus instead.

The latter accepts tentatively, and then, as an answer to Socrates' question about science/knowledge, he gives a sequence of sciences: geometry, astronomy, (musical) harmony, calculation ('λογισμός'), all sciences already mentioned by Socrates, adding cobblery and other practical techniques ('τέχναι') [5] (146c-d).

Socrates reproaches him for his 'generosity' (146d4-5), previously praised by Theodorus as one of his quality (144d3). Indeed, the young boy gave many examples, when one only answer was asked, what is **science**[6].

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[4] As a matter of fact, at the very beginning Socrates makes clear the inquiry will be as much about Theodorus (and Theaetetus), his character as well as his mathematics and his teaching (144d8-145b1). The polysemy of some terms essential for the text, particularly '*λόγος*' and '*ἐπιστήμη*', makes its translation difficult, but also sometimes its understanding itself.

[5] A mixing Socrates immediately denies by changing Theaetetus' 'techniques' into 'sciences' (146d8).

[6] It is already an example of the main subject in this mathematical part, incommensurability. In the introduction of Plato's *Statesman*, where the same characters appear as in another dialogue of Plato, the *Sophist* supposed to happen the next day after the discussion told in *Theaetetus*, Socrates berates vivaciously Theodorus when he puts on the same level politics and philosophy. And indeed, the latter has to admit there is no proportion between them (257a9-b8).

Here, Theaetetus put into the same pot geometry and cobblery, sciences (ἐπιστῆμαι) and techniques (τέχναι) (146d7-8). Socrates then corrects him simply by changing the incorrect wording, as he will proceed several times afterwards. As a matter of fact, he simply asks him if when he says 'cobblery' ('σκυτικήν'), he means 'science of making the shoes' ('ἐπιστήμην ὑποδημάτων ἐργασίας') instead (146d7-8), or 'carpentry' ('τεκτονικήν') he means in fact 'science of making wooden furnishings' ('ξυλίνων σκευῶν ἐργασίας', 148e1-2)). Moreover, he incidentally replaces a plural 'techniques' ('τέχναι') by a singular 'science' ('ἐπιστήμη') (146d7-8). Theaetetus' generosity causes him not only to give an

Then, suddenly, Theaetetus says Socrates' explanations about his question 'what is science?', reminds him of a similar problem. As he and a friend of him, Socrates' namesake[7], were thinking about Theodorus' lesson, they asked themselves a problem in connection with this lesson. It is the beginning of the so-called 'mathematical part' of *Theaetetus*.

In order to facilitate the understanding of the mathematical presentation, we may sometimes give deliberately some anachronistic transcription. Such an approach is not without difficulties[8], since it may hide some conceptual problems[9]. We will thereby give in each case a transcription according to ancient Greek geometry and inside the known limitations of the mathematics in this period.

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irrelevant answer, but also to mix things which have no common measure. Interestingly, there is a similar anecdote of Diogenes the Cynic, but directed against Plato's 'generosity' ([Laerce1999], VI, 2, 26).

[7] It is the name given in Plato's book. According to a tradition maybe from the Middle Age, he is usually referred to by the translators as 'Young Socrates' to distinguish him from Socrates, the philosopher. However we will stand here with his name 'Socrates', the context giving which Socrates it is the name.

[8] As an illustration we can consider a very simple example, the commutativity of the multiplication (i.e. for two integers $m$ and $n$, the product of $m$ by $n$ is equal to the product of $n$ by $m$). It is a pure triviality for us moderns, though in Euclid's *Elements*, it is given in the 16th proposition of book VII, using a large part of the theory of ratios of integers.

[9] An example of such a difficulty may be found in the supposed proof of irrationality through the so-called '*anthyphairesis*' construction (cf. infra, §5). It is usually done for only one case and moreover using modern algebra and modern symbols (cf. for instance [Waerden1950], p. 143-146).

## 3. Generalities about the text

The importance of the ‘mathematical part’ of *Theaetetus* is due not only it is the oldest text about Greek mathematics and irrational magnitudes, but also that it is a writing about the working and the practice of mathematicians in classical Greece (5th century BCE).

The young Theaetetus begins with an account about a lesson given by Theodorus about mathematical irrationality. The latter showed the incommensurability with the unit of the sides of the squares of areas *3* feet, *5* feet, … till *17* feet. In modern words, we may say he showed the irrationality of $\sqrt{3}$, $\sqrt{5}$, …, $\sqrt{17}$. Nevertheless, according to Theaetetus, Theodorus made use of a given unit of length, the Greek ‘foot’ (‘ποδιαία’)[10], and moreover he used it to compute areas[11].

Theaetetus and his comrade Socrates[12] decided, after the lesson, to work together on what Theodorus taught them. Firstly they claim there are evidently an infinite number of such incommensurable magnitudes and they decide to try to get a general characterization of them instead of a case by case proof (‘κατὰ μίαν ἑκάστην’) as Theodorus proceeded. Then they remark these questions may be translated into a geometric language through the association of each and every integer to a planar figure.

Firstly they divide all the integers, into two classes. The first contains the numbers product of an integer by itself[13]. The other class contains all the other integers or, as Theaetetus explains, the ones it is not possible to write as such twice the product of an integer, for example *3* and *5* and all the numbers only product of ‘a greater integer by a les or a les by a greater’ (148a2-4). Thus the figure representing a number in the first class, product of an integer by itself, is a square of side this latter integer. The figure representing the one in the second class is a rectangle whose sides are given by the couple of unequal integers whose product is equal to the former number[14].

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[10] About 30 cm.

[11] Some translators (for instance Fowler 1996, McDowell 1973 and more recently Chappell 2004) do not hesitate to translate it as ‘square foot’. It is mathematically correct, but textually problematic. It is also the choice of Canto-Sperber (cf. [Canto1993]) in her French translation of *Meno* (82c-84b). Nevertheless they usually bring the existence of a problem to the readers’ attention, even if the reason given is pure speculation: a supposed confusion by ancient Greek geometers between units of length and area.

[12] It is of course not the philosopher, but a friend of Theaetetus of the same age, cf. *supra*, §2.

[13] For instance the ones like *9,…* but Theaetetus does not specify.

[14] Though it is not explicitly said, it includes the product whose one factor is the unit. For the only possibility to write a prime number as a product of integers, is a product of itself by the unit. Thus, according to Theaetetus account, the unit is a particular integer which is not

A difficulty is hidden in this classification, the question of his effectivity. As long as only the first odd numbers are considered[15], it is easy to know in which class they belong, for they have few divisors and moreover many of them are even prime number[16]. It is still true for small numbers, possibly using a table of squares[17]. But in the general construction of Socrates-Theaetetus, it is not so easy to say when a number is a square or not[18]. As a matter of fact, it is a consequence of Theodorus' lesson beginning at 3 and ending at 17 and considering the odd numbers each after the other ('case by case').

There is a second much often overlooked point in Theodorus' lesson. The mathematician is working only with areas of squares and lengths as sides of such squares[19], both given in the same unit (the foot). In consequence there

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evident according to ancient Greek mathematics. The unit, and in some texts, even *2*, were sometimes not considered as integers. Thus, according to definition 2 of book VII in Euclid's *Elements*, the integer ('ἀριθμός') is defined as a 'multitude composed of units' ('τὸ ἐκ μονάδων συγκείμενον πλῆθος').
However, according to Socrates-Theaetetus' method, it is always possible to associate a rectangle to a given integer. But conversely, several rectangles may be generally associated to the same integer. For instance, to the integer 6, it is possible to associate either a rectangle of sides *1* and *6* or a rectangle of sides *2* and *3*.

[15] For a justification, cf. *infra*, note 58.

[16] Between 3 and 17 there are only two odd non-prime integers (9 and 15).

[17] Many of such tables are found on Babylonian tablets, but not in old Greece. Either they disappeared because written on some fragile material, unlike the clay tablets of Mesopotamians, or maybe they did not use them.

[18] One possibility is to find it by successive approximations, another by finding its divisors, but both methods are time-consuming.

[19] In modern terms, integers and their square roots.

are only two kinds of magnitudes, for lengths as well as areas[20]: the ones equal to a whole number of units i.e. of feet and the others incommensurable to the unit i.e. the foot. Thus the result taught by Theodorus appears as the following: the length of the sides associated to the square of area a whole integer *n* of feet is either a whole number, when *n* is a 'perfect square'[21], or incommensurable to the unit-foot otherwise.

After attending the lesson, Theaetetus continues, 'it occurred to' him and his friend Socrates, to divide integers into two classes (147d-e). The first corresponds to the geometrical squares for numbers equal to the product of an integer by itself ('ἴσον ἰσάκις'[22]). The second correspond to geometrical rectangles with unequal sides, called by the boys 'elongated integers' ('προμήκη αριθμόν'[23]) (148a3). Then in connection with these classes two other classes appear: according to whether the area belongs to the first or the second class, its side belongs either to the first or the second new class[24].

Then Theaetetus goes on, there are two possibilities. Either a line belongs to the first class, thus it is a 'real' line, for it is called a 'line' ('μῆκος'), or it belongs to the other class and is something like a quasi-line, for it is not commensurable ('οὐ συμμέτρους') with the previous ones, the 'real' lines (148b1).

In modern mathematical terms, it means the first class consists of integers, the second class of irrationals, more precisely they are square roots of these very integers which are not perfect squares. It is a direct generalization of what Theodorus taught the boys about the sides of some squares of given area stopping at the square of area 17 feet.
Indeed, Socrates and Theaetetus gave a general definition, valid for any length $\sqrt{2}$, $\sqrt{3}$, … A first difference is the square root of *17* has no special role.

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[20] Cf. *supra*, note 11.
[21] A 'perfect square' is an integer raised to the power 2 or in other terms it is the square of an integer. For instance $9 = 3^2$ is a perfect square.
[22] Indeed, the Greek sentences seem extremely awkward, reflecting the youth and the inexperience of the two boys.
[23] Same remark as above.
[24] The meaning is the two new classes are defined as follows: the first class is formed of the sides of the square whose area is a square number i.e. belonging to the first class previously defined; the second class is formed of the sides of the square whose area is an 'oblong number' i.e. belonging to the second class previously defined. An element of the first new class is called 'length' ('μῆκος'), in the second new class they are called 'powers' ('δυνάμεις'). Once again, the construction is awkward. Instead of our explanation, the geometrical square associated to any rectangle is not specified, though it is geometrically the most important step (Aristotle remarks it is equivalent to the definition of a geometric mean (*Métaphysics* III, 996b18-21; *On the Soul* II, 2, 413a13-20)).

Moreover, the boys left what can be called the 'metrical geometry' for the foot, which is both a length and area unit is left out. Nevertheless, as in Theodorus lesson, nothing is said about some possible intermediaries, both commensurable magnitudes ('σύμμετρα μεγέθη') as they are defined in definition 1 of the book VII of Euclid's *Elements*[25], but non integers i.e. corresponding to the lengths whose ratio to the unit[26], is equal to the ratio of two prime integers.

Nevertheless, contrary to Theodorus' way of teaching, the boys have left the physical measures of areas given in foot-units. Their statement is no more about really drawn figures, or at least possibly drawn by hand. It is about integers and abstract magnitudes. The drawings are replaced by some form of reasoning, at least partially. Such mathematics may be considered more sophisticated and closer from the one as conceived by Plato, as for instance in book VII of the *Republic*, than the mathematics in Theodorus' lesson[27]. This already justifies Socrates' flattering approval of the boys' work given a little further, at the end of Theaetetus' account[28].

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[25] According to definition X.1, two magnitudes are commensurable when they are measured by the same measure; here one of these magnitudes is a unit of measure. Then definition X.3 gives the definition of 'rational' ('ῥητή') line. Firstly, an arbitrary line is called 'rational' ('ῥητή'); then any line commensurable with it is called 'rational' ('ῥητή'). Moreover Euclid adds an intermediary between rational and irrational lines, what he calls 'rational in power'. Then, taking the unit line as the arbitrary 'rational' line, any line considered by Socrates-Theatetus (and *a fortiori* by Theodorus in his lesson) is 'rational' in Euclid's sense.

[26] Or in Socrates-Theatetus' language, the ratio to any (real) 'length'.

[27] Cf. also Pappus' criticism of Theodorus' "metrical" conception. Though it seems to be intended to be explanatory, the rest of Pappus' text is not so clear. It is maybe an addendum of some commentator or translator who did not completely understand the question. Nevertheless, its general sense is pretty clear ([Pappus1930], §11, p. 74).

[28] We do not study here this account given by Theaetetus about his common work with his friend, the other Socrates. It will be the subject of a next article (OfmanWIP).

## 4. Socrates-Theaetetus' "result"

Though as previously said[29] we do not study the work of the two boys, there is a point in Theaetetus' account which matters here. It is his mention as in passing the sides of the squares whose areas are non-perfect integers and the ones of perfect integers are incommensurable (148b). Rewritten in modern terms, it means the following:

'**Socrates-Theaetetus' result**':
'The square root of an integer is rational if and only if the integer is a perfect square'.

**Remark.** It is important to note, for the confusion has been often done[30], 'Socrates-Theaetetus' result' is completely different of the following:
'The square root of an integer $n$ is an integer if and only if $n$ is a perfect square'.

For the latter, written in symbolic form:
$\sqrt{n} = N$ is equivalent to: $n = N^2$
is a complete triviality resulting from the very definition of square root, or in geometrical terms, it results from the very definition of the surface of a square with respect to its sides.
Rather, 'Socrates-Theaetetus' result' means the following equivalence:

$\sqrt{n} = p/q$ (with $p$ and $q$ integers) is equivalent to: $n = r^2$ for some integer $n$.

Or its contrapositive better suitable for a future proof by *reductio ab absurdum*:
$n$ is **not** a perfect square if and only if f**or any integers** $p$ and $q$, we have:
$\sqrt{n} \neq p/q$ i.e. $nq^2 \neq p^2$ [31].

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[29] Cf. *supra*, note 26.
[30] It is difficult to escape such confusion, since in Theaetetus' account, only integers and irrationals are considered. In the same way, in *Meno* (82b-85b), by beginning with a square of sides two feet long, Socrates is avoiding fractions and makes sure he has only to use integers in the computations for doubling a square.
[31] Indeed, in both cases, it is the implication from left to right which matters.
- In the first case, it means:
$\sqrt{n}$ rational entails there is an integer $r$ such that $n = r^2$ (the inverse is trivial for if $n = r^2$, the square root of $n$ is $r$, an integer *a fortiori* a rational).
- In the second case (the contrapositive), it means:
If the integer $n$ is not a perfect square, then there are integers $p$ and $q$ such that: $\sqrt{n} = p/q$.
The converse is evident since it means :
there are no integers $p$ and $q$ such that: $\sqrt{n} = p/q$ (then $n = p^2/q^2$) then $n$ is not a perfect square (and this entails the possibility to choose $q = 1$).

The difficulty here is it is impossible to proceed directly, case by case, for $p$ and $q$ are **any** integers!

## 5. Validity of Socrates-Theaetetus' reasoning.

Now we will check the validity of the boys' reasoning. It is undeniably needed it for any mathematical statement. The same is still true for an inquiry supposed, as Socrates said, to get the same degree of certainty than a mathematical proof, looking to the truth and excluding anything only plausible or probable (162e).

'Socrates-Theaetetus' result' is often considered as the same as the proposition 9 of book X of Euclid's *Elements*. It is not totally true, for the former is a particular case of the latter[32]. It was already noted in the Antiquity. For instance Pappus insisted in his commentary of the book X on their differences ([Pappus1930], 10-11, p. 73-75)[33].

There is no shortage of proofs for this result. However there is a difficulty in Theaetetus' account[34]: according to it, Theodorus stops indeed when he studied, **one by one** ('κατὰ μίαν ἑκάστην') the incommensurability 'in length with the unit of one foot'[35] ('μήκει οὐ σύμμετροι τῇ ποδιαίᾳ') of the sides of some squares of areas 3 feet, 5 feet, …, 17 feet[36]. Therefore he definitively did not give the general proof of the 'Socrates-Theaetetus' result'.
*A fortiori*, it is quite unbelievable two very young boys may have done such an achievement, just after attending a lesson given by a well-known mathematician unable to obtain it. This common sense position[37] is confirmed by the text which contains nothing close from any proof, not even the necessity of a proof.
From Theaetetus' account, such property seems to be a natural consequence of Theodorus' lesson, flowing without a sound with the perfect gentleness of a stream of oil[38]. It may seem to have been surreptitiously introduced by Theaetetus and his comrade, but they are totally unaware of it.
It is such a serious problem modern commentators find it hard solving it, and they disagree about the solution. Some of them claim Theodorus did not give any proof during his entire lesson, trivializing the whole 'mathematical

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32 As a matter of fact, from a strictly mathematical point of view, 'Socrates-Theaetetus' result', as usually understood, would be closer from proposition VIII.24 of the *Elements* than proposition X.9 (cf. also, note 25).
33 M. Burnyeat is rightly complaining that modern commentators did not consider this important point ([Burnyeat1978], note 60, p. 507).
34 147d.
35 Once again an awkward way to indicate the irrationality of such lengths.
36 Cf. supra, note 11.
37 It is also the common view of authors as different as Heath, Knorr, Szabó, Caveing or Burnyeat.
38 It was how Theodorus was calling Theaetetus' extraordinary ease to learn (144b5).

part'[39]. Hence, one of Plato's longest mathematical texts would be a series of platitudes. This runs counter to the whole tradition from the Antiquity to the modern times, as well Plato's use of mathematics in his other books[40].

Others believe Plato's purpose in his book is to pay tribute to Theaetetus as a colleague and friend. Thus he puts in Theaetetus' mouth some theory the latter will discover several years later.

It is losing the sight of his young friend Socrates, the nickname of the philosopher. In Plato's text, he is a co-author in this extraordinary discovery, though he never appears in the tradition concerning the theory of irrationality, which is always attributed only to Theatetus. Moreover how to understand Plato would consider the absence of a proof in a mathematical account as laudable? Socrates a little further insists on the essential characteristic of proof in mathematics, and Theodorus agrees warmly with it[41].

But the most decisive objections are the following.

The first is internal. Socrates repeatedly insists on the lack of seriousness of children (for instance, 148c1-4; 165a, d; 166a; 168d-e; 169b9; 200a), or even teenagers' works[42], which are mere 'child's play' when compared to adults' works.

Within this framework, it is really unconceivable to display two young boys discovering one of the most crucial results of Greek geometry which the greatest mathematicians of his time were unable to prove[43].

On the other hand, a reader in Plato's days, contemporary of Theaetetus' works, would definitively know all the hard work needed and the difficulties to solve the problem. Hence it is unreasonable to believe he may uncritically agree to such historical distortions, especially in a book claiming to follow the mathematical rigor of reasoning[44]. And it is even more unreasonable its author may think it, within a polemical environment where the many philosophical

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39 For instance A. Szabó ([Szabo1977], p. 63-65).

40 One only have to remember the mathematical cosmogony in *Timaeus* or the presentation of the 'nuptial number' in *Republic* (VIII, 546b-c).

41 'but you do not advance any cogent proof whatsoever; you base your statements on probability. If Theodorus, or any other geometrician, should base his geometry on probability, he would be of no account at all.' ([Plato1921]) ('ἀπόδειξιν δὲ καὶ ἀνάγκην ούδ' ἡντινοῦν λέγετε ἀλλὰ τῷ εἰκότι χρῆσθε, ᾧ εἰ ἐθέλοι Θεόδωρος ἠ ἄλλος τις τῶν γεωμετρῶν χρώμενος γεωμετρεῖν, ἄξιος ούδ' ἑνὸς μόνου ἂν εἴη.' (162e)).

42 At 168e1-3, Socrates demands Theodorus to enter into his analysis of Protagoras' doctrine to avoid it to be accused being 'a play with mere young people' ('ὡς παίζοντες πρὸς μειράκια'). A little further, he again cautions Theodorus against a discussion which may be accused to have a 'childish form' ('παιδικόν (…) εἶδος') (169b9-c1).

43 It leads some commentators or historians, as for instance J. Itard (cf. [Itard1961], p. 34), to claim Plato was a poor connoisseur of the mathematics of his time (cf. *supra*, note 39).

44 As M. Burnyeat rightly emphasizes, it is on the utmost importance taking into account the background of the readers at Plato's time ([Burnyeat1978], p. 491), for they were the ones for whom Plato was writing.

schools were ferociously fighting each other[45], and the slightest error would be used against him by the many opponents of the Academy.

[45] Let us remember the anecdote reported by Diogenes Laertius: to make fun of Plato's definition of a man as 'a biped without feathers', Diogenes the Cynic, during some Plato's lecture, hurled a featherless chicken at him ([Laerce1999], VI, 2, 40).

## 6. Six criteria to be fulfilled by Theodorus' demonstration

As we have seen, the absence of any indication on the method used by the mathematician[46] in Theatetus' account, leaded some commentators to conclude that the whole lesson was just trivial[47]. Such a claim is rather paradoxical since its supporters use its supposed unimportance for characterizing the latter as a series of banalities. Such an extreme point of view is so implausible (cf. *supra*, note 37), most historians consider valuable the search of the method used by Theodorus to prove the irrationality of the square roots of *3, 5, …* till *17* which is in accordance with Plato's text[48].

Its omission by Plato concerning his reader, or Theaetetus concerning Socrates, is easily understood inside the framework of Greek education[49]. The problem as well as its proof was well known by the people, at least the reasonably well-educated people[50] making up Plato's public. Hence they

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[46] At least concerning the construction of the squares of areas *3, 5, …, 17* feet *per se*. Hence many commentators wonder if there is some constructions and which ones. See for instance J. Anderhub, W. Knorr, H. Schmidt, A. Szabó, van der Waerden … *contra* T. Heath who claims the text does not entail such constructions ([Heath1921], p. 203, note 2). W. Knorr is more ambiguous since he agrees only partially with Heath ([Knorr1975], p. 74).However, there is mention of *this* construction of squares, as Theaetetus telling only vaguely Theodorus spoke about ('περί') of squares (or of their sides, depending of the translation). Anyway, among historians who think this construction was indeed given, there are strong disagreements about the method used for the construction (cf. for instance [Waerden1963], p. 142-143). According to the proof we propose, such a construction does not really matter. However it has an important role to play for the understanding of Theodorus' mathematics and Plato's appraisal. If it is not of direct interest here, it will be crucial in a work in progress ([OfmanWIP]). As for Heath's argumentation, it is principally directed against any attempt to trivialize Theodorus' lesson.

[47] For instance, for a supporter of this thesis, cf. [Szabo1977], p. 66; at the contrary, M. Burnyeat rejects such a conclusion and claims the demonstration itself is of no help for the understanding of the text ([Burnyeat1978], p. 505).

[48] Not without some skepticism sometimes (e.g. [Burnyeat1978], p. 505).

[49] Cf. *supra*, note 42.

[50] The interest of Greek people toward mathematics would certainly amaze our contemporaries. Thus there is a remark, quoted in [Laerce1999] (III, 11) about the impossible invariance of the even and the odd when a unit is added included in a comedy of Epicharmus of Kos (mid-6th century BCE; Plato, in *Theaetetus* 152e5, refers to him as someone at the height of comedy). In Aristophanes' *Birds* (999-1009), one finds a joke about squaring the circle. It confirms the high regards in which Greeks and especially Athenians held mathematics as highlighted in Plato's books. It explains the ease Plato's characters understand Socrates' mathematical examples. It is also in line with the offensive nature of the Athenian in the *Laws* towards the ones, including the other characters, ignorant of mathematical irrationality, calling them 'guzzling swine' (819d-820c). The insult is admittedly softened for he includes himself in these very people, which obviously is not the case. Lastly one notes also the enthusiasm of all the young boys to learn mathematics with Theodorus, and his popularity among them, as emphasized at the beginning of *Theaetetus* (143d, sq.).

could not be ignorant of the chronology and what was predating the general irrationality theory.
It is Theaetetus himself who, according to the tradition, will be its author some years afterwards. This very theory, with some slight modifications, is also the one we find in Euclid's *Elements*. Thus, neither Theaetetus for Socrates, nor Plato for his readers, need to worry about reminding it. Moreover, as we will see in the next paragraph, the sequence of integers studied by Theodorus is a strong hint on his method. It makes sense only if the proof is based on some property of odd squared integers (cf. *infra*, §7.1).

Mathematicians and historians have conceived a lot of models for Theodorus' method. The most common and successful among modern interpreters is based on what is called '*anthyphairesis*' by transliteration of a Greek term. Briefly it is a generalization to magnitudes of the so-called 'Euclid's algorithm'. This algorithm, defined on integers, gives for any couple of integers, their Greatest Common Divisors[51].

Such a hypothesis is not without many difficulties. One is its complexity and the need to use some infinite process analogous to what is called in modern mathematics 'infinite descent' not seen anywhere in extant Greek mathematical texts. Moreover, because of its very complexity and length, it is totally impossible to have been taught during one mathematical lesson (and even several indeed), though this is explicitly stated in Plato's book[52].
Given the many suggested demonstrations, one would easily think the most difficulty is to determine the one actually used by Theodorus. However, the opposite is true when the conditions required by Theaetetus' account are taken into consideration. Actually, according to Plato's text, any such demonstration needs to satisfy six conditions[53] given hereinafter[54]. Unfortunately, none of

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[51] Cf. for instance [Waerden1963], p. 145. Roughly speaking it consists in subtracting as many times as possible the smallest (let it be $b$) from the greatest (let it be $a$), so that the rest $r$ is smaller than $b$, and then to iterate this time with $b$ and $r$. It is possible to continue as long as the smallest does not measure (i.e. is not a divisor of) the greatest, since at this step it comes to an end. Such a conclusion (after a finite number of steps) is true if and only if $a$ and $b$ are commensurable (*Elements*, propositions X.2 et X.3).
[52] We will give a more detailed analysis in a next work in progress (cf. also the course on history of Greek mathematics, part III, in line at http://www.math.jussieu.fr/~ofman). To argue Plato as any author is totally free to write what he wants and to present the characters and their discussions totally arbitrarily is simply a non-sense. On one hand, such a consequence would make this part purely fictional, so that it becomes worthless as far as mathematics is involved. And more important, it would be once again forgetting the historical background within which the Old Academy was developing (cf. supra, end of previous paragraph, in particular notes 44 and 45).
[53] There are indeed seven conditions. The last one follows from the very end of Theaetetus' account when he switches to cubic roots. But as previously said, it is outside the present article (cf. *supra*, note 28).

them verify all of them, most verifying one or two at best. For example the proof by *anthyphairesis* satisfies condition ii) [55] but none of the others, not even condition iv). Indeed it was probably already known and used by the old Pythagoreans, according to many (late) testimonies[56], but inside an arithmetical framework and/or to get some approximations.

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[54] W. Knorr already noted the need of such a textual control. He came to the same conclusion: no previous demonstration was consistent with the text (cf. [Knorr1975], p. 96-97). However, our conditions are stricter than his. It means if a proof verifies our listed conditions, it is admissible by Knorr, but the inverse is not true. In particular there is nothing similar to our condition v), though it is essential in order that it is something else than a mere tale (ib., p. 193).

[55] This is emphasized by its supporters (cf. e.g. [Waerden1963], p. 143).

[56] It is called into question by many historians inside the general suspicion against testimonies of late Antiquity about Pythagoras and its school, especially everything concerning mathematics.

The six conditions necessary by Plato’s text:

i) Beginning at 3, not at 2;

ii) To be done case by case (‘κατὰ μίαν ἑκάστην’), not though a general result;

iii) To explain the stop at 17;

iv) To be consistent with mathematical knowledge at the period of Theaetetus’ account;

v) To be able to be taught during one lesson for very young boys;

vi) To induce a wrong generalization to the whole of the integers.

## 7. A proof verifying all these six criteria

a) As odd as it may seem, no one considered the question of Theodorus' sequence of integers, for the answer seemed evident.

However Theaetetus does not report this very sequence, but what one may call an 'abbreviation': *3, 5, ..., 17*. Thus it begins at *3*, the next term is *5* and it stops at *17*, but nothing is known on the integers between *5* and *17*. Moreover there are hot discussions about the last integer *17*, whether it is included in the list studied by Theodorus and does the latter say anything about it? Historians of mathematics are deeply divided about the answer[57].

Conversely, since the Antiquity, commentators wonder why Theodorus begins at *3* instead of *2*, for the square root of *2* is irrational and it is simpler to prove it[58].

Most of modern historians accept Hieronymus Zeuthen's explanation, with or without some reservation[59]. Plato leaves it out for the irrationality of the square root of *2* was well-known.

It is clear they are right that the result is a very old one. But it is not so sure it is the reason of its absence in Theodorus' lesson. And indeed, the demonstration of the irrationality of the square root of *2* gives much more: the general question about the irrationality of the square roots of integers is then

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[57] W. Knorr remarks rightly any sequence of integers not including *1, 2 and 4,* but *3* et *5*, and stopping à *17* may be suitable here.

[58] In a previous article ([Ofman2010]), we gave a demonstration using only very old and elementary results, and moreover consistent with textual testimonies, contrary to the usual proposed proofs. Though it is not needed to understand Theodorus' method in his mathematical lesson, it makes it simpler and more consistent (cf. *infra*, note 60). It is an additional argument supporting it as the first demonstration of mathematical irrationality. Conversely, from the point of view of the *anthyphairesis*, the integer *2* is the simplest case, which adds another argument against it as Theodorus' method. And indeed,

- To claim the irrationality of the square root of *2* is also proved by *anthyphairesis* would entail to change dating of its apparition in Greek mathematics. This would in turn entail new difficulties and even begging the results. Since Theodorus was contemporary of this result, if he is not the author of the discovery, its importance in mathematics would be severely reduced, as well as the interest of his lesson. Moreover, whether or not he is the author, the absence of *2* in Theodorus' sequence is hardly comprehensible. And most important, such a method goes directly against the extent textual testimonies.
- On the other hand, one may assume the case of *2* was already known and proved in a different manner, so that Theodorus would be the first to use the *anthyphairesis* to prove the irrationality of other square roots. But it does not make easier to explain *2* is missing. Indeed, the simplest case for the proof of irrationality by an *antiphairetic* construction is precisely the case of the square root of *2*. A teacher would certainly begin by it, and afterwards only he would give the more complex generalization. In any case, he should at least mention the universal property of the method. In the absence of such a hint, Plato's reader may rightly question the credibility of the whole book (cf. *supra*, note 44).

[59] Sometimes it is taken as a last resort explanation, because no others are available, for instance [Burnyeat1978], p. 502-503.

reduced to the case of odd numbers. I is even enough to prove they are (or are not) equal to a ratio of two odd integers. In the first case they are rational, in the second they are irrational[60].

Thus the whole problem concerning the rationality/irrationality of the square roots of integers is to answer the following question:
for any odd integer $n$ whether it exists (or does not exist) two odd integers $p$ and $q$ such that: $\sqrt{n} = p/q$ [61].

As a direct consequence, Theodorus' sequence is easy to deduce: *3, 5, ..., 17* is simply the natural sequence of the first odd integers (excluding the unit) and ending at *17*.
Moreover, according to the usual interpretations, it is understood Theodorus avoided perfect squares integers. However this is certainly not the case according to Theaetetus' account. True, translated in modern terms for brevity, he says the mathematician showed the square roots of *3* and *5* are irrational. But then he considered 'one by one in turn up to seventeen' the other integers where he stopped ('καὶ οὕτω κατὰ μίαν εκάστην προαιρούμενος μέχρι τῆς ἑμτακαιδεκάποδος; ἐν δὲ ταύτη πως ἐνέσχετο')[62]. Thus, clearly, none was excluded.
Now for such a natural sequence of integers, it is clear enough no more explanation was needed, either for Socrates of for Plato's reader.

It therefore remains to be seen when the square roots of these integers are or are not rational.

b) The proof we propose here is based on a very simple result already known by old Pythagoreans about the odd perfect square integers[63]. Moreover it is

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[60] For more details, cf. [Ofman2010], p. 118.
[61] Though it can be deduced from most methods of proof, it is an evident consequence of the one in [Ofman2010]. However it is important to notice that the whole question of irrationality is reduced to the case of odd numbers does not mean we know the case of even numbers. For instance, as a consequence of the demonstration in the previously cited article, one gets immediately the irrationality of the square root of *6*. However since $12 = 4 \times 3$, the square root of *12* is equal to twice the square root of *3*, so that it is rational (or irrational) if and only if *3* is rational (or irrational).
[62] According to the usual interpretations, all the integers where studied, except of *4*, *9* and *16*. However, according to our interpretation the only perfect square in the sequence is *9*, and it is included in the number studied by Theodorus.
[63] See for instance, Plutarch, *Platonic Question*, II, 24, 1003f. Jean Itard remarked the interest of this result ([Itard1961], p. 34-36), and it was adopted later by several historians including M. Caveing, W. Knorr, J. Vuillemin. As a matter of fact, the only contemplation of arrays of odd squared numbers leads to such a deduction, and the proof is extremely simple, even in inside a primitive arithmetic. According to Itard, it was even possible such a result had been already known by Babylonian or Egyptians calculators (ib., p. 34).

strongly connected to Theodorus' sequence[64] as define in the previous paragraph. It is the following result:

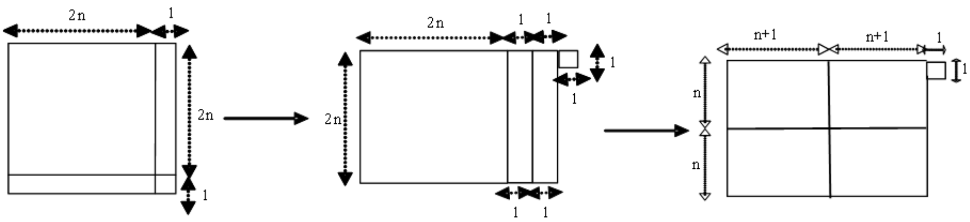
1. **'The remainder result**': The remainder of the division by *8* of a squared odd integer is *1*.

- In particular, if it is not the unit, it needs to be greater than *8*.
- In modern terms it means that any odd number to the square is a multiple of *8* plus *1*, or in more scholarly terms it is equal to *1 modulo 8*.

This result follows from the:
**Very simple remark**: when *8* is added to any integer *n*, the remainder in the division by *8* does not change. Thus, it does not change either by adding any multiple of *8* (i.e., *n* et *n+8m* divided by *8* have the same remainder).

The proof is very simple. In modern algebraic form, it is enough to write:
$(2n+1)^2 = 4n^2 + 4n + 1 = 4n(n+1) + 1$.
Since either $n$ or $(n+1)$ is even, then $n(n+1)$ is even, so that $4n(n+1)$ is a multiple of *8*. Hence, from the 'very simple remark' above, $4n(n+1) + 1$ divided by *8* has the same remainder than *1* (divided by *8*) i.e. *1*.

But the geometrical proof is as simple, and it may be as follows[65]:




## 2. The use of the 'Remainder result'

It remains to be seen how this result may be used, though it is about the integers only and not about (what we call) rational numbers (cf. *supra*, remark of §4).

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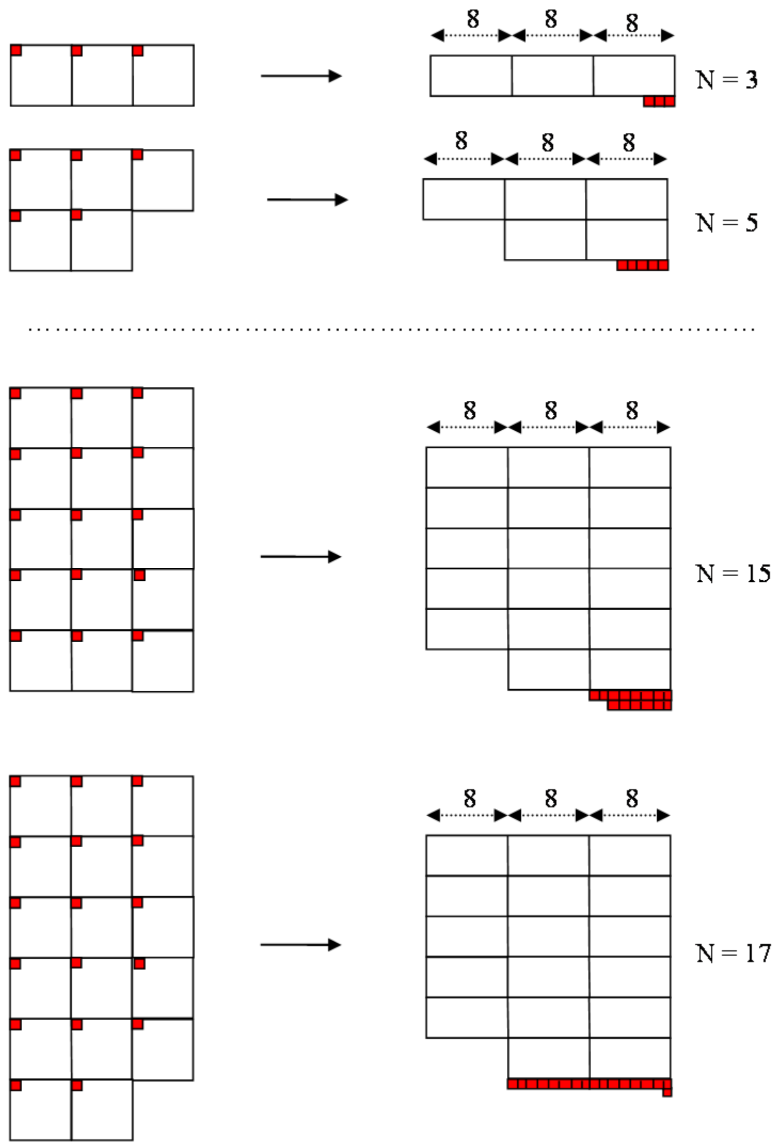
[64] Once again, one need emphasize Plato's readers did not have the same difficulties we have about Theodorus' method, especially because the very sequence told them how he proceeded.
[65] This result is assigned to the old Pythagoreans by Proclus (cf. [Proclus1992]), in particular in his commentary on the first propositions of book I of Euclid's *Elements*. The knowledge of the identity: $(m+n)^2 = m^2+2mn+n^2$ (*m* and *n* integers), is needed for the oldest known proofs of particular cases of Pythagoras' theorem. It was already in use in Babylonia in the 2nd millennium BCE as shown by some tablets of the Schøyen Collection (cf. [Friberg2007], p. 42-51).

As reminded at the beginning of the paragraph (cf. *supra*, note 60), the square root of an odd integer is rational if and only if there exists odd integers $p$ and $q$ such that: $\sqrt{n} = p/q$.
If there are such integers, the square root of $n$ is rational; otherwise it is irrational.
This equality entails:
$n = p^2/q^2$
or
$nq^2 = p^2$.

Since $q$ is odd, the 'Remainder result' gives:
$nq^2 = n\,(8k+1) = 8nk + n$.
Since $nq^2 = p^2$ and $p$ is odd, we may apply the same result both to $nq^2$ and $8nk + n$, so that: the remainder of $8nk + n$ divided by $8$ is equal to $1$. From the 'very simple result', it is the same for $n$, thus we get finally:

the remainder of $n$ divided by $8$ is equal to $1$. (*).

**Remark**. The above algebraic proof is evidently anachronistic. We give thereafter a geometrical one where the result is *read* on the drawing.




The colored parts in the above diagram represent one unit (in Theodorus' language a square of a foot), the white parts are areas multiple of *8*. The left to right arrow means the passage from the left figure to the right one. It transforms truncated squares (i.e. squares minus one unit, hence 'gnomons'[66]) into rectangles whose one side is *8* units. It is obvious on the drawing the remainder of any multiple of *3* divided by *8* is *3* (not *1*), of any multiple of *5* divided by *8* is *5* (not *1*), and so on.

[66] A sort of set-square used in many fields, in particular for astronomical calculations.

Let us consider the array with on the left part, the integers $n$ of Theodorus sequence and on the right part, their remainders in the division by $8$:

| $n$ | reste de la division de $n$ par $8$ |
|---|---|
| $3$ | $3$ |
| $5$ | $5$ |
| $7$ | $7$ |
| $9$ | $1$ |
| $11$ | $3$ |
| $13$ | $5$ |
| $15$ | $7$ |
| $17$ | $1$ |

The first three integers ($3, 5, 7$) do not verify (*), thus according to the 'Remainder result' their square roots ($\sqrt{3}, \sqrt{5}, \sqrt{7}$) are not rational.
The first difficulty is with $9$, but $9$ is a perfect square ($9 = 3^2$), so that its square root (namely $3$) is an integer thus rational.

Likewise, the three following integers ($11, 13, 15$) do not verify (*), thus their square roots ($\sqrt{11}, \sqrt{13}, \sqrt{15}$) are once again irrational.

Now the remainder of $17$ divided by $8$ is $1$. But this time $17$ is not a perfect square, so that the boys conclude naturally its **square root is irrational**.
However, though one may obviously conclude $\sqrt{17}$ **is not an integer**[67], it is impossible to say anything about its **rationality or irrationality**.

Hence Theodorus stopped here and did not conclude. Either because he wanted his students to find the result by themselves (as claimed by A. Szabó[68]), or more probably because he wanted they attended the following course[69].
But such a way of teaching is not safe. As a matter of fact the young boys understood Theodorus' work may be generalized to an infinity number of

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[67] Indeed $17$ is greater than $16 = 4^2$ and less than $25 = 5^2$, and there is no integer between $4$ and $5$.
[68] [Szabó1977], p. 92.
[69] Teachers were paid according to their audience: when Socrates is speaking with teachers, generally the so-called 'sophists', money matters are never very far. For instance at the very beginning of the dialogue with Theodorus, the first subject is money, more precisely Theaetetus' money (140c6-d3). Indeed further in the dialogue, when he is discussing some theses of Protagoras, a friend of Theodorus, Socrates reminds the former demanded 'huge fees' ('μεγάλων μισθῶν', 161d).

integers[70] (147d), though his 'case by case' solution was no more valid in this situation.

It is also another reason for Socrates' compliment[71] to the two young boys (Socrates and Theaetetus) a little further (147e8, 148b7).
However, Theodorus' demonstration[72] does not expose the main problem at *17*, so his stop seems making no sense. Theodorus 'I do not know why, stopped here' says Theaetetus (147d7)[73]. Hence the boys will conclude the alternative given by Theodorus is true still in the generalized situation:

- Either it is possible to apply the 'Remainder result'.
- Or when it is not possible, there is a new alternative:
  - Either the integer is a perfect square (as *9*) and its square root is rational (as an integer).
  - Or it is not a perfect square (as *17*) and then its square root is irrational.

Here the boys completed Theodorus because the result seems obvious. Hence the mathematician did not say anything about its square root, and certainly not it was impossible to obtain an answer[74]. He just left it open, and rightly this seems surprising for the boys.

The above demonstration based on the 'Remainder result' verifies the six criteria in the previous paragraph. Actually:

- It explains the beginning à 3 and the stop at 17.
- It is done case by case and uses only very old arithmetical knowledge, Theodorus probably adapted to the square roots of integers.
- It is easily done in a one hour lesson, a reasonable length for a mathematical lesson for young boys.

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[70] It is already evident as soon as $\sqrt{2}$ is known to be irrational, since for any integer $n$, we have: $\sqrt{(2n^2)} = n\sqrt{2}$ is irrational. This equality was used by Socrates in *Meno* (83a-c) to show the young slave how doubling a given square. Thus it was a very ancient knowledge. Moreover it is evident as soon as the figure is drawn.
[71] Cf. *supra*, note 25.
[72] His demonstration uses only integers and not more general rational magnitudes. It is rightly emphasized by Pappus in his commentary of Euclid *Elements* ([Pappus1930], §10, p. 73; cf. also *supra*, note 32).
[73] It is our translation of 'ἐν δὲ ταύτῃ πως ἐνέσχετο'. As a matter of fact the translation is a much disputed problem. For instance Michel Narcy translated it by 'quelque chose l'a arrêté [là]' ('something stopped him [here]' ([Platon1994]). But according to some other commentators it may be a pure random stop, though this is not actually consistent with the Greek text.
[74] At least according to the method Theodorus used in his lesson.

- It explains moreover the hasty conclusion of the young boys and a statement they were not able to prove (cf. *supra*, end of §5).

## 8. Conclusions

### a) The aporetic nature of Theodorus' demonstration

According to usual interpretations, the 'mathematical part' of the *Theaetetus* accounts on one hand for Theodorus' discoveries on irrationality and on the other hand, it gives prophetically a fundamental result of the future theory of the irrationals as found in Book X of Euclid's *Elements*. This theory is indeed traditionally related to Theaetetus at a later period when he was become a mathematician. This makes difficult to relate the 'mathematical part' to the rest of Plato's book.

Indeed, the exegetical tradition claims the book is fundamentally aporetic, and every try to find a solution ends in a series of failures. As a matter of fact, much to the chagrin of Theodorus, of Theaetetus[75] and probably also of the reader, all these conclusions are meticulously refuted one by one by Socrates. So that it is not easy to understand Plato's (or Socrates') own position on the successive definitions proposed by Theaetetus.

On the contrary, the mathematical account would announce one of the main successes in Greek mathematics, the theory of irrationality. Thus, this part instead to enlighten the reader would be confusing[76]. However according to our analysis, this part of *Theaetetus* is consistent with the rest of the book.

Admittedly Theodorus' results give the solution in a lot of cases, or in modern language one would say it gives the correct answer in 80% of cases[77].

From an *effective* point of view the process is efficient (in the sense for instance of modern algorithmic), and Theodorus (as well as Theaetetus and his friend Socrates) can be proud of his own work. Nevertheless, from a mathematical point of view, it is unable to characterize the rationality/irrationality of the square roots of integers, in the sense of Socrates' question (147c1, 146d3-4, 146e7-9, 147e1, 148d4-7), though the two boys believe it does (147e1). As previously with Theaetetus' answer to Socrates' 'what is the science?', we get many answers but not the right answer. Moreover later Greek mathematics will not take this approach. The

[75] Cf. e.g. 157c4-6, 161a4-9, 165d1.

[76] For instance B. van der Waerden, though he is not usually very rigorous when commenting Plato's texts (for instance [Waerden1950], p. 141), claims the 'mathematical part' looks like 'an afterthought (…) which does not really fit there' (ib., p. 166 ; see also p. 142).

[77] Or the probability to get an (correct) answer is greater than 80%. It means when a number is randomly selected, one has more than 8 out of 10 to know if it is irrational. For example, in the sequence from 1 to 17, Theodorus' method (plus what was known about even integers, cf. *supra*, note 58) fails only in one case (the last one).

‘Remainder result’ will be left in favor of the theory of proportions and relative prime numbers[78].

Mathematically Theodorus’ method is actually an apory. A change of method was needed to get out of it. The same holds for Socrates’ question ‘What is science?’, for at the end of the book, one does not get the answer.

But conversely, this parallelism shows the long inquiry about science in *Theaetetus* is not only an apory. The last definition given by Theaetetus, a ‘true opinion with a reasoning (‘λόγος’)’ is likely more relevant than the list given at the very beginning of the dialogue (cf. *supra*, §2). Continuing on this parallelism, though the last definition is definitively not the definition of ‘the science’, it fits nevertheless well for most ‘sciences’[79]. Then, the main stumbling block of the inquiry is the oversight of a needed division between ‘the sciences’[80]. The method of definitions through divisions will be a central point in the *Sophist* presented as a sequel of *Theaetetus*, as well as in the *Statesman*[81].

### ii) **Plato’s criticism of Theodorus**

Plato’s criticism is done on two levels. On one hand, against the way Theodorus makes mathematics. According to Theaetetus’ account, he does not do it through reasoning but drawings (‘ἔγράφε’) (147d2), and moreover on a ‘case by case’ basis (‘κατὰ μίαν ἑκάστην’, 147d6). But as soon as the infinite appears, such a method is doomed to failure. According to this account, Theodorus’ lesson is even well short of the intellectual abilities of two young boys. The latter reason directly on all the integers. They abandon completely physical measures used by Theodorus, namely the foot[82].

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[78] Once again it is needed to emphasize the effectiveness of Theodorus’ method even in comparison with the later one. To get a result for any given number, using the former one, one have only to divide it by *8*, or to divide it successively by *2*, depending if it is odd or even. Through the latter method one needs to get the decomposition of the number into his prime factors, a long time-consuming process. Hence, when one wants to know if a square root of a given integer is rational or not, it is reasonable to use Theodorus’ method (today it would be the first to be implemented into computers to get a solution). Actually in more than 80% of cases, we get the answer. However the point is, one will not **always** get it. Then, but only then, after a failure of Theodorus’ method, the latter one may be used.

[79] As a matter of fact, it may be suitable for almost all ‘sciences’ except of precisely mathematics, the science taught by Theodorus and learnt by Theaetetus. The exclusion of mathematics would result from precisely the same errors found in Socrates-Theateteus reasoning (cf. the analysis of the end of the ‘mathematical part’, *supra*, §5).

[80] Though Socrates had already imposed a division between ‘sciences’ and ‘handicraft techniques’, Theaetetus had previously gathered (cf. *supra*, note 5).

[81] In another book Socrates claims to be a ‘lover of divisions (‘διαιρέσεων’)’ (*Phaedra*, 266b4).

[82] It does not mean Theodorus did not know the global problem and the difficulties arisen by the infinity. The criticism is about his presentation of mathematics.

Plato is no less critical of Theodorus' way of teaching. He used drawings instead of reasoning. But, as most drawings, they hid the difficulties[83]: anything impossible to be graphically represented simply does not exist. Its mathematical consequences are as follows.
There are three different possible disjoint cases to consider:

- The square root is an integer.
- The square root is not an integer but is rational.
- The case the square root is irrational.

However only two cases (the first and the last) are considered by the young boys.
As a matter of fact, they are actually the only possible ones, hence they are the only ones graphically representable.
And yet the mathematical method does not condone such an approach. It has to be proved through a careful demonstration, there are indeed only two cases i.e. the second one is impossible. But only an abstract reasoning may do it[84]. Moreover, in this problem, such a proof is extremely difficult to elaborate[85]. Indeed, the figures are misleading the boys leading them to believe as evident a property which will literally change Greek mathematics[86].

### iii) **Consequences for the understanding of *Theaetetus***

This judgement of Plato on Theodorus' mathematics, and thus on Theaetetus' one, at least when he was his student, entails a global change of perspective.
Rather than on one hand praising Theodorus supposed to have been Plato's master, and on another hand paying a tribute to Theaetetus, his supposed friend and colleague, as claimed in the usual interpretations, Plato blames them and their mathematics.

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[83] I the same way as clothes conceal bodies (165a1).
[84] More generally any *reductio ad absurdum* needs such an abstract proof, for it is impossible to draw the impossible. It does not mean diagrams cannot be used in such kind of proofs, but certainly not like Theodorus used them in his lesson.
[85] It is connected to what is called in modern mathematics 'Gauss' theorem' or the 'fundamental theorem of arithmetic'. Its meaning is any integer can be represented 'uniquely' as a product of powers of prime numbers (the 'unicity' has to be understood with some evident restrictions). The difficulties to overcome were all the more considerable when the undeveloped state of Greek arithmetic in the last years of the 5th century BCE is taken in consideration, the time when the dialogue is supposed to have taken place.
[86] The general theory of irrationality is essentially given in book V of Euclid's *Elements* traditionally assigned to Eudoxus. Such a theory makes sense only outside of the framework of commensurable magnitudes (in modern terms irrational numbers), hence after some general theory of mathematical irrationality had been elaborated.

It may be worth emphasizing our mathematical study of this part of *Theaetetus*[87] agrees with Michel Narcy's point of view developed in the introduction of his translation ([Platon1994], especially pp. 40-69). Starting from a purely textual and philosophical analysis, the author proceeds by comparison to other texts of Plato on mathematics, in particular in *Republic*.
It is somewhat an exterior confirmation of our analysis based on the mathematical questions arisen from this part.

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[87] As the reader has certainly noticed, we carefully avoided any philological questions. We will return to them in a future work.

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